IXI - MOSTI

Study of the ILS certification process
Confidence limits and probabilities.

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1 Introduction

1.1 Context and purpose

The purpose of this study is to explain some parts of the DERA report DERA/WSS/WX1/CR 980799/2.3. ≪ILS Certification Requirements≫ (Ref. 2). We mainly study section 3.3 ≪Confidence Limits for Sequential Tests≫ p.28-34 and section 2 of the appendix D, p.69-72.

The assertions of this last section have been used in page 3 of the note AWOG/8-WP/10 of 30/04/2001 (Ref. 7) :

≪*Two possible certification schemes are defined. If the design MTBO is at least twice the requirement (providing some confidence that the system will meet the requirement), a sequential test plan compiled to provide at least* 60% *confidence shall be performed...*

Although termed ≪60% *test*≫ *it should be noted that the confidence level that can be achieved will be between* 88% *(accepting the system after 1 year without outages) and* 64% *(accepting the system with 8 outages).*≫

These assertions are the main justifications for choosing the use of the sequential test plan with a consumer's risk equal to 40%.

The sequential test plan is preferred to a classical test (time fixed) because generally the decision process is more speedy. The associated methodology is attractive because it allows as the observations arrive:

- to decide acceptation,
- or to decide rejection,
- or to continue to observe the equipment, when the two first decisions cannot be taken.

1.2 Glossary

Estimator : All statistic used to approximate an unknown parameter.

Estimation : Value of the estimator.

ILS : Instrument Landing System.

MTBO : Mean Time Between Outages.

objective MTBO : Value of the MTBO that the equipment has to achieve to be qualified.

Outage : Service interruption of the equipment.

R.R.V. : Real Random Variable.

Statistic : Function of the observations regarded as a R.R.V.

True MTBO : Real MTBO of the equipment; generally this value is unknown.

1.3 Notations

θ : Parameter describing the possible values of the MTBO of the equipment.

θ_1 : objective MTBO of the equipment.

θ_0 : d times the objective MTBO, where d is a fixed value.

Standardized time : time in hours divided by $\theta_1 = \theta_0/d$ where θ_1 is chosen equal to the objective MTBO

$t' = \dfrac{t}{\theta_1}$: standardized time.

$\theta' = \dfrac{\theta}{\theta_1}$: standardized parameter.

t_{A_i} : acceptance time so that we accept equipment if no more than i outages occur. (After i outages the result of the sequential test leads up to accept the hypothesis that the MTBO of the equipment is equal to d times the objective MTBO).

t_{R_i} : rejection time, so that we reject the equipment if at least i outages occur.

t'_{A_i} : standardized acceptance time associated to i outages, $t'_{A_i} = \frac{t_{A_i}}{\theta_1}$ (cf p.10).

t'_{R_i} : standardized rejection time associated to i outages, $t'_{R_i} = \frac{t_{R_i}}{\theta_1}$ (cf p.10).

$\mathbb{P}\left(\left(i; t'_{(k)}\right); \theta'\right)$: probability that i outages have occurred in a total test standardized time $t'_{(k)}$ without terminating the test (we cannot accept or reject), when θ' is the value of the standardized parameter (the true MTBO is $\theta'\theta_1$). This probability is called : continuation probability.

$\mathbb{P}\left(\left(i; t'_{A_i}\right); \theta'\right)$: probability that the sequential test ends with the acceptance of the equipment at the time t'_{A_i}, when θ' is the value of the standardized parameter. This probability is called acceptance probability.

$\theta_{L,\gamma,i}$: $100\,(1-\gamma)\,\%$ Lower confidence limit after acceptation with i outages. So we have $\mathbb{P}[\theta > \theta_{L,\gamma,i}$ / the equipment is accepted after i outages$] = (1-\gamma)$.

t'_0 : Standardized value of the maximum total test time (truncation).

i_0 : Maximum number of outages allowed during the sequential test (truncation).

$N\left(t'_{(k)}\theta_1\right)$: Number of outages that occur during the observation time $t'_{(k)}\theta_1$ (Notation $R_{t'_{(k)}\theta_1}$ of Ref 12).

1.4 Reference documents

1. **MIL-HDBK-781A** : Handbook for reliability test methods, plans, and environments for engineering, development, qualification, and production. 1 April 1996. Department of Defense (USA).

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DERA/WSS/WX1/CR980799/2.3.

3. **Sequential Analysis**: Wald A., 1947, John Wiley & Sons, New York.

4. **Sequential Life Tests in Exponential Case**, Epstein B. and Sobel M., 1955, Annals of Mathematical Statistics, Volume 26, pp. 82-93.

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6. **Sequential Analysis : tests and confidence intervals**, D. Siegmund, 1985, Springer-Verlag.

7. **All weather operations group (AWOG) co-ordination meeting**, report of the project team on certification, 30/04/2001.

8. **Sequential Tests of Statistical Hypotheses,** B.K. Ghosh, 1970, Addison-Wesley publishing company.

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10. **Confidence Limits on MTBF for Sequential Test Plans of MIL-STD 781**, C.Bryant and J.Schmee, 1979, Technometrics, Volume 21, pp 33-42.

11. **Sequential Analysis, Direct Method**, L.Aroian, 1968, Technometrics, Volume 10, pp 125-132.

12. **Rapport concernant la possibilité d'utiliser le MTBO théorique dans la procédure de qualification des équipements ILS**, IXI/SDO/00/P093/N1, (IXI-MOSTI), 18/01/2001.

13. **Méthodes de calcul numérique**, Nougier, 1983, Masson.

1.5 Report presentation

First, we detail the calculation of the true MTBO lower confidence limits after acceptance ($\theta_{L,\gamma,i}$). Then, we explain the calculation procedure of the confidence probability after acceptance when the acceptance has occurred after i outages.

Confidence probability means : probability that the true MTBO is higher than a given value. In practice this given value is generally equal to the objective MTBO value.

Some calculations are used to obtain these probability values in accordance with different plans.

These numerical results allow us to give our point of view on the procedure.

2 Confidence intervals

2.1 Introduction

There are many ways to estimate a parameter. One of then, is the calculation of a numerical value. This value is not very useful if it is not possible to get an idea of the precision of this estimation. So, we prefer to give an interval with some assurance that the true parameter θ lies within the interval.

Let $X_1, X_2, \ldots, X_n$ be n independent identically distributed RRV with density $x \longmapsto f(x, \theta)$ when θ is the true value of the parameter.

Definition 1 *Let $B_1 = T_1(X_1, \ldots, X_n)$ and $B_2 = T_2(X_1, \ldots, X_n)$ be two statistics such that*

$$\mathbb{P}[B_1 < \theta < B_2] = 1 - \alpha \quad ,$$

then $[B_1; B_2]$ is an interval of probability $1 - \alpha$ associated to θ.

Definition 2 *All realization $[b_1; b_2]$ of an interval of probability $1 - \alpha$ is called a $100(1 - \alpha)\,\%$ confidence interval of θ.*

Example *Assume that for $i = 1, \ldots, n$ X_i has a Gaussian distribution $\mathcal{N}(\theta; \sigma^2)$ where σ^2 is known. The maximum likelihood estimator of the mean is $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. The quantity $\dfrac{\overline{X} - \theta}{\sigma/\sqrt{n}}$ will be normally distributed with zero mean and unit variance and we have, for example*

$$\mathbb{P}\left[-1,96 < \frac{\overline{X} - \theta}{\sigma/\sqrt{n}} < 1,96\right] = 0,95 \tag{1}$$

$$\Longrightarrow \quad \mathbb{P}\left[-1,96\frac{\sigma}{\sqrt{n}} + \overline{X} < \theta < 1,96\frac{\sigma}{\sqrt{n}} + \overline{X}\right] = 0,95 \quad . \tag{2}$$

We get here $B_1 = -1,96\frac{\sigma}{\sqrt{n}} + \overline{X}$ *and* $B_2 = 1,96\frac{\sigma}{\sqrt{n}} + \overline{X}$.

We assume that $n = 4$, $x_1 = 1,2$, $x_2 = 3,4$, $x_3 = 0,6$, $x_4 = 5,6$ *and* $\sigma = 3$. *So* $\overline{x} = 2,7$. *Then the 95% confidence interval of* θ *is*

$$\left[\overline{x} - \frac{3}{2}(1,96)\,;\overline{x} + \frac{3}{2}(1,96)\right] \quad . \tag{3}$$

We obtain $[-0,24; 5,64]$.

2.1.1 Interpretation

Usually we write

$$\mathbb{P}\left[-0,24 < \theta < 5,64\right] = 0,95 \quad . \tag{4}$$

It is a notation and it is important to understand its meaning.

Only notation (2) is exact. It tells us that the probability of the random interval $[B_1; B_2]$ is equal to $0,95$.

This means that if we repeat 100 times the calculation of the interval (3) with successive samples of n observations, interval which would be different for each sample, the interval contains the true value of the parameter θ in 95% of those statements.

The next figure shows the result of computing 50 percent confidence intervals of the parameter θ for 15 samples of size 4 actually drawn from a normal population $\mathcal{N}(0;1)$, assuming the variance **unknown**.

The intervals are shown as horizontal lines above the θ-axis, and, as expected, about half of them covers the true value 0 of the parameter θ.

2.2 General method

The two previous examples are specific, because we can obtain a function of the observations and the parameter θ such that its distribution is independent of θ. This kind of function is called "pivotal".

2.2.1 Method

Generally we apply the following method. Let $\widehat{\theta_n} = Y_n$ be an estimator of θ and $y \longmapsto g(y,\theta)$ its density when θ is the true value of the parameter. Let $\gamma \in]0;1[$; we may find two numbers h_1 and h_2 such that

$$\mathbb{P}_\theta\left[Y_n < h_1\right] = \int_{-\infty}^{h_1} g(y,\theta)\,dy = \frac{\gamma}{2} \tag{5}$$

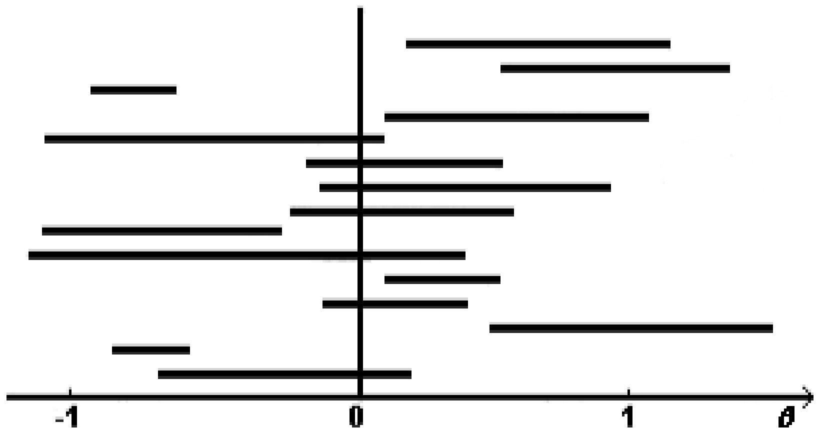
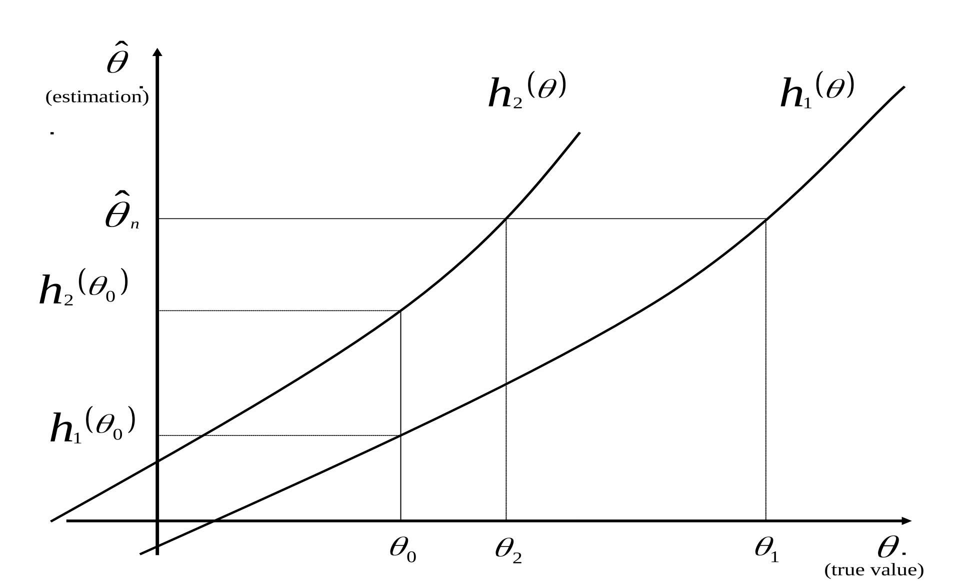


Figure 1: 50% confidence interval



$$\mathbb{P}_\theta \left[Y_n > h_2 \right] = \int_{h_2}^{+\infty} g\left(y, \theta\right) dy = \frac{\gamma}{2} \tag{6}$$

The interval of probability $1 - \gamma$ is not unique. We can decide to change $\frac{\gamma}{2}$ in (5) and $\frac{\gamma}{2}$ in (6) by γ_1 and γ_2 such that $\gamma_1 + \gamma_2 = \gamma$. When we have no more information we choose $\gamma_1 = \gamma_2 = \frac{\gamma}{2}$.

When θ varies, the quantities h_1 and h_2 vary also and can be viewed as two functions of θ. Then we can write

$$\mathbb{P}_\theta \left[h_1\left(\theta\right) < Y_n < h_2\left(\theta\right) \right] = \int_{h_1(\theta)}^{h_2(\theta)} g\left(y, \theta\right) dy = 1 - \gamma. \tag{7}$$

Then h_1 and h_2 may be plotted against θ.

A vertical line through any chosen value θ_0 will intersect the two curves h_1 and h_2. The projection of the intersection points on the $\widehat{\theta}$-axis give the limits $h_1(\theta_0)$ and $h_2(\theta_0)$ between which θ will fall with probability $1-\gamma$.

$$\mathbb{P}_{\theta_0}\left[h_1(\theta_0) < Y_n < h_2(\theta_0)\right] = 1-\gamma.$$

For a given sample of size n, we get an estimation $\widehat{\theta_n}$. The horizontal line through $\widehat{\theta} = \widehat{\theta_n}$ intersects h_1 and h_2 in two points such that their projection on the θ-axis denoted by θ_1 and θ_2 respectively are the bounds of the confidence interval.

If we denote by Θ_1 and Θ_2 the RRV, with values θ_1 and θ_2, we get

$$\Theta_2 < \theta < \Theta_1 \Longleftrightarrow h_1(\theta) < Y_n < h_2(\theta)$$

hence

$$\mathbb{P}\left[\Theta_2 < \theta < \Theta_1\right] = \mathbb{P}_\theta\left[h_1(\theta) < Y_n < h_2(\theta)\right] = 1-\gamma.$$

Then $]\theta_2;\theta_1[$ is a $100(1-\gamma)\,\%$ confidence interval for θ.

2.2.2 Remarks

1- It is sometimes possible to determine the limits θ_2 and θ_1 of a given estimate without actually finding h_1 and h_2. The points θ_2 and θ_1 are such that $h_1(\theta_1) = h_2(\theta_2) = \widehat{\theta_n}$. And θ_1 is the value of θ for which

$$\int_{-\infty}^{\widehat{\theta_n}} g(y,\theta)\,dy = \frac{\gamma}{2} \tag{8}$$

and θ_2 is the value of θ for which

$$\int_{\widehat{\theta_n}}^{+\infty} g(y,\theta)\,dy = \frac{\gamma}{2}. \tag{9}$$

If the left-hand sides of these two equations can be given explicit expression in terms of θ and if the equation can be solved for θ uniquely, then those roots are the $100(1-\gamma)\,\%$ confidence limits for θ.

2- The previous method may be used with a sample of discrete R.V. and a discrete estimator of θ. The integrals (8) and (9) become sums.

Let $g(y,\theta)$ be the probability that $\widehat{\theta}$ has the value y, the equations cannot generally be solved exactly and the method gives confidence interval higher than $100(1-\gamma)\,\%$.

3- This method could be applied, more generally, to every statistic such that its distribution is a function of θ and the calculations (5) and (6) are feasible.

3 Scope of this study

For Cat.III localizer, the objective MTBO is a MTBO greater than 4000 hours. So we should test

$$H_0 : \theta > 4000 \text{ against } H_1 : \theta < 4000.$$

The sequential probability ratio test only works with simple hypotheses, that is to say hypotheses where the considered parameter value is unique. It is a common use to test

$$H_0 : \theta = \theta_0 = 8000 \text{ against } H_1 : \theta = \theta_1 = 4000.$$

Then when the true value is too close to 4000, the hypothesis H_0 is rejected.

The next calculations are developed with the use of standardized times. The parameter used to standardize the time is not unique but generally the parameter used is θ_1. The standardized value 1 (the lower test MTBO) is associated to θ_1 and the standardized parameter $\theta' = \frac{\theta}{\theta_1}$ is associated to θ.

Remark : *the standardized time is denoted t'.*

A graph may be used to describe a sequential test plan with the number of outages on the y-axis and the standardized time on the x-axis.

We recall here, some parts of the rapport IXI/SDO/00/P093/N1 (Ref. 12).

Let α the producer's risk and β the consumer's risk. These two values are fixed. We calculate

$$A = \frac{(1-\beta)(d+1)}{2\alpha d} \quad \text{and} \quad B = \frac{\beta}{1-\alpha} \quad \text{where} \quad d = \frac{\theta_0}{\theta_1}.$$

We have in the plan (time, number of outages) two lines

$$\begin{aligned} r &= a + bt, \\ r &= c + bt \end{aligned}$$

(see details page 16 of Ref 12).

It is possible to give the decision rule as a function of time. Denote

$$D = \frac{1}{\theta_1} - \frac{1}{\theta_0} > 0 \quad h_0 = -\frac{\ln B}{D},$$

$$h_1 = \frac{\ln A}{D} \qquad s = \frac{\ln d}{D}.$$

The possible decisions of the sequential probability ratio test (Wald), which is denoted by $S(A, B)$, are

- continue to test if $rs - h_1 < t < rs + h_0$,

- accept H_0 if $t \geq rs + h_0$,
- accept H_1 if $t \leq rs - h_1$.

Moreover, this test can be truncated if we decide to end the test after the time value t_0 or if we decide to end the test after i_0 outages.

If no decision has been reached before t_0 or i_0 then we decide:

- to accept H_0 if the time t_0 is reached but not i_0,
- to accept H_1 if i_0 outages have occurred during a time smaller than t_0.

Moreover i_0 is the smallest integer such as

$$\frac{\chi^2_{2i_0,\alpha}}{\chi^2_{2i_0,1-\beta}} \geq \frac{\theta_1}{\theta_0} = \frac{1}{d}$$

and

$$t_0 = \frac{\theta_0 \; \chi^2_{2i_0,\alpha}}{2}.$$

Remark: *these values were denoted n_0 and T_0 in the Ref.12.*

To obtain the decision rule when we use standardized times, we denote

$$h'_0 = -\frac{\ln B}{1-\frac{1}{d}} \quad h'_1 = \frac{\ln A}{1-\frac{1}{d}},$$
$$s' = \frac{\ln d}{1-\frac{1}{d}} \qquad t' = \frac{t}{\theta_1},$$

r the number of outages.

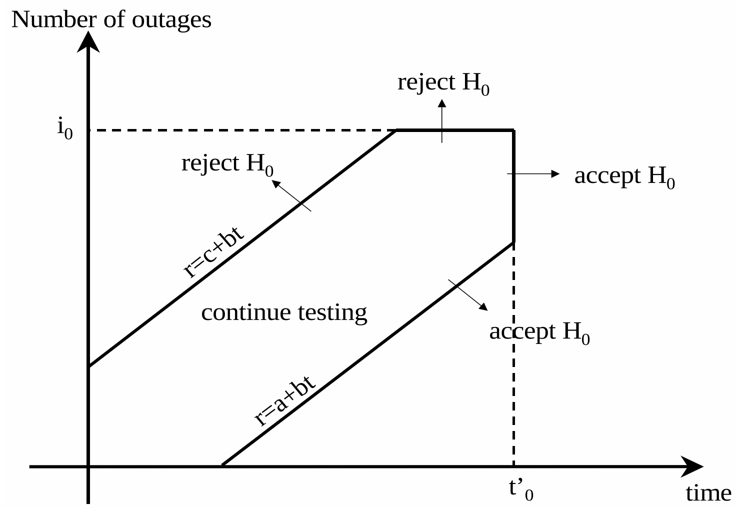
Then t'_0 the standardized time of truncation is equal to

$$t'_0 = \frac{d}{2}\chi^2_{2i_0,\alpha}.$$

The decision rule of the test $S(A, B)$ with the standardized time is :

- if $t' \geq rs' + h'_0$ we accept H_0,
- if $t' \leq rs' - h'_1$ we reject H_0,
- if $rs' - h'_1 < t' < rs' + h'_0$ we continue to observe the equipment.

We use the same truncation than the truncation used in the DERA report (Ref 2, p28), but it is different from the truncation used in the AWOG note (Ref 7). In this case the time associated with the truncation has been delayed



(the time of end of the test is higher than t'_0). Nevertheless the equations of the confidence lines (acceptance or rejection) are the same :

$$U_0 \text{ (boundary of acceptance) is given by } \quad r = \frac{t' - h'_0}{s'},$$
$$U_1 \text{ (boundary line of rejection) is given by } \quad r = \frac{t' + h'_1}{s'}.$$

We call (t', i) a point of the continuation zone which can be reached after i outages detected during the total standardized time t'.

We call $\mathbb{P}((i, t')\,; \theta')$ the probability that the test continues, when θ' is the true value of the parameter and i outages are detected during the total standardized time t'.

We call t'_{A_i} the **standardized acceptance time** so that we accept the equipment if no more than i outages occur in $t'_{A_i} \times \theta_1$ hours. It is a standardized time of end of test with acceptance.

Then $\mathbb{P}\big((i, t'_{A_i})\,; \theta'\big)$ is the probability that the test ends with acceptance of the equipment at the time $t'_{A_i} \times \theta_1$ when θ' is the true value of the parameter.

Let t'_{R_i} the **rejection standardized time** such that the equipment is rejected with i outages or more at or before the time $t'_{R_i} \times \theta_1$ hours.

Example *Figure 4 – with this sequential test plan, if the objective MTBO of the tested equipment is θ_1, we accept the equipment if we observe one outage only during the time $\theta_1 t'_{A_1}$. We reject this equipment if we observe 4 outages during the time $\theta_1 t'_{R_4}$.*

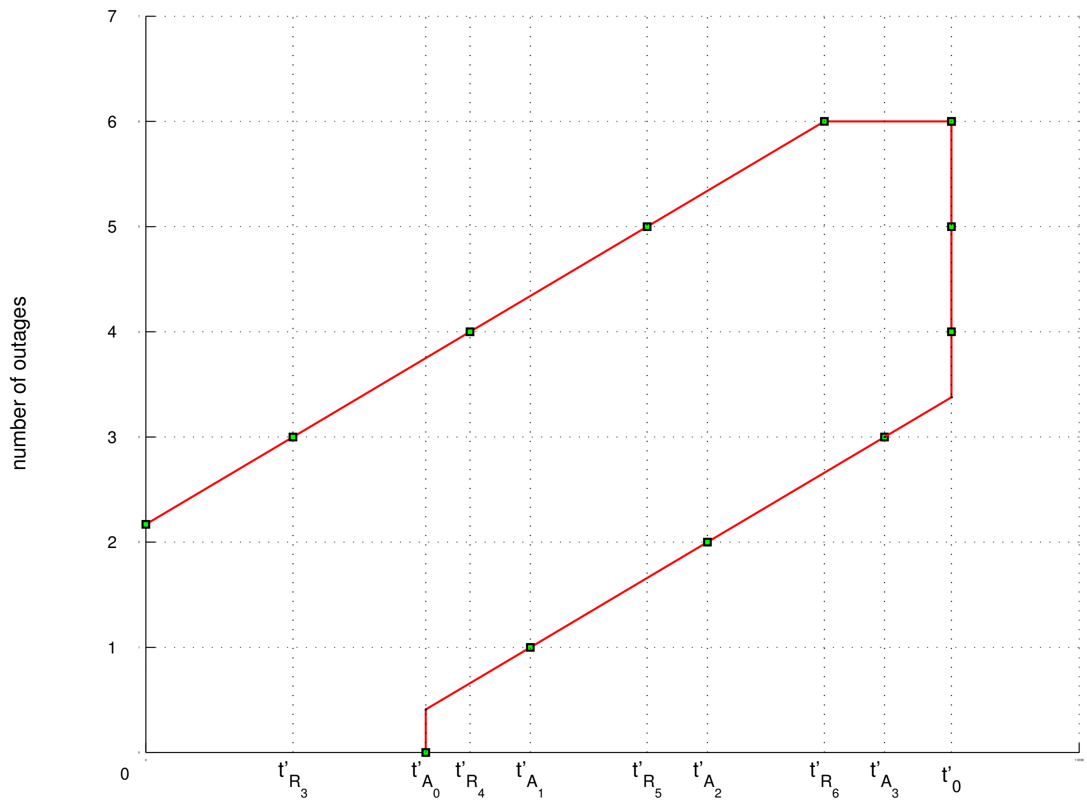


Figure 2: 4- Position of end of test time

4 Mathematical appendix

4.1 Confidence limit

To calculate the confidence limits of the parameter θ, it is necessary to evaluate the acceptance and the continuation probabilities $\mathbb{P}((i,t')\,;\theta')$ for some values of t'.

4.1.1 Acceptance and continuation probabilities

Let $\left(t'_{(k)}\right)_k$ be a sequence of standardized termination times ordered such that $t'_{(0)} = 0$ and $t'_{(j)} = t'_0$. Denote for $l = 1...j$

$$\Delta_l = t'_{(l)} - t'_{(l-1)}.$$

So we obtain

$$\left]0, t'_{(k)}\right] = \cup_{l=1}^k \left]t'_{(l-1)}, t'_{(l)}\right] \text{ and } \sum_{l=1}^k \Delta_l = t'_{(k)}.$$

Let $\left(t'_{(k)}, i\right)$, $k \leq j$, be a point on the continuation zone or on the acceptance boundary. We calculate the probabilities $\mathbb{P}\left(\left(i, t'_{(k)}\right);\theta'\right)$ for all the possible values of θ'. For a fixed value of the standardized parameter θ', the probability that δ_l outages occur in the interval $\left]t'_{(l-1)}, t'_{(l)}\right]$ is equal to

$$\mathbb{P}\left(\delta_l \text{ outages in } \left]t'_{(l-1)}, t'_{(l)}\right];\theta'\right) = \mathbb{P}\left(N\left(t'_{(l)}\theta_1\right) - N\left(t'_{(l-1)}\theta_1\right) = \delta_l;\theta'\right).$$

Moreover $N\left(t'_{(l)}\theta_1\right) - N\left(t'_{(l-1)}\theta_1\right)$ has a Poisson distribution with parameter $\dfrac{\Delta_l}{\theta'}$. So, we have

$$\boxed{\mathbb{P}\left(\delta_l \text{ outages in } \left]t'_{(l-1)}, t'_{(l)}\right];\theta'\right) = e^{-\frac{\Delta_l}{\theta'}} \frac{\left(\frac{\Delta_l}{\theta'}\right)^{\delta_l}}{\delta_l!}.}$$

Now we look to $(\delta_1, ..., \delta_k)$, positive integers such as $\sum_{l=1}^k \delta_l = i$ and not leading to a termination before $t'_{(k)}$. Then the probability that they are i outages during the time $t'_{(k)}$ with for each l, δ_l outages in the interval $\left]t'_{(l-1)}, t'_{(l)}\right]$ without terminating the test before $t'_{(k)}$ is equal to

$$\begin{aligned}
& \mathbb{P}\left((\delta_1, ..., \delta_k)\,, \text{ no termination before } t'_{(k)};\theta'\right) \\
= \; & \mathbb{P}\left(\forall l = 1...k,\ \delta_l \text{ outages in } \left]t'_{(l-1)}, t'_{(l)}\right];\theta'\right) \\
= \; & \mathbb{P}\left(\forall l = 1...k,\ N\left(t'_{(l)}\theta_1\right) - N\left(t'_{(l-1)}\theta_1\right) = \delta_l;\theta'\right).
\end{aligned}$$

By independence of the increments of N , we have

$$\begin{aligned} & \mathbb{P}\left((\delta_1,...,\delta_k)\,,\ \text{no termination before } t'_{(k)};\theta'\right) \\ = & \prod_{l=1}^{k} e^{-\frac{\Delta_l}{\theta'}} \frac{\left(\frac{\Delta_l}{\theta'}\right)^{\delta_l}}{\delta_l!} \\ = & \left[\prod_{l=1}^{k} e^{-\frac{\Delta_l}{\theta'}} \left(\frac{1}{\theta'}\right)^{\delta_l}\right]\left[\prod_{l=1}^{k} \frac{(\Delta_l)^{\delta_l}}{\delta_l!}\right] \\ = & e^{-\frac{t'_{(k)}}{\theta'}} \left(\frac{1}{\theta'}\right)^{i} \prod_{l=1}^{k} \frac{(\Delta_l)^{\delta_l}}{\delta_l!}. \end{aligned}$$

Then the probability that i outages have occurred in a total test time $t'_{(k)}$ for a fixed value of the standardized parameter θ' is

$$\begin{aligned} & \mathbb{P}\left(\left(i,t'_{(k)}\right);\theta'\right) \\ = & \sum_{S} \mathbb{P}\left((\delta_1,...,\delta_k)\,,\ \text{no termination before } t'_{(k)};\theta'\right) \\ = & e^{-\frac{t'_{(k)}}{\theta'}} \left(\frac{1}{\theta'}\right)^{i} \sum_{S}\prod_{l=1}^{k} \frac{(\Delta_l)^{\delta_l}}{\delta_l!}, \end{aligned}$$

where S denotes all possible k-uples $(\delta_1,\ \delta_2,\ldots,\ \delta_k)$ such that $\sum_{l=1}^{k}\delta_l = i$ and the test is not terminated before $t'_{(k)}$.

Let us denote

$$c'\left(i,t'_{(k)}\right) = \sum_{S}\prod_{l=1}^{k} \frac{(\Delta_l)^{\delta_l}}{\delta_l!}, \tag{10}$$

we get

$$\mathbb{P}\left(\left(i,t'_{(k)}\right);\theta'\right) = e^{-\frac{t'_{(k)}}{\theta'}} \left(\frac{1}{\theta'}\right)^{i} c'\left(i,t'_{(k)}\right). \tag{11}$$

Let us denote

$$c\left(i,t'_{(k)}\right) = c'\left(i,t'_{(k)}\right) i! \left(\frac{1}{t'_{(k)}}\right)^{i}. \tag{12}$$

Remark : *With this modification, we get coefficients* $c\left(i,t'_{(k)}\right)$ *with values belonging to a shorter interval than the interval of the coefficients* $c'\left(i,t'_{(k)}\right)$.

Thus we have shown that the probability that i outages have occurred in a total test time $t'_{(k)}$ and the test is not ended is

$$\boxed{\mathbb{P}\left(\left(i,t'_{(k)}\right);\theta'\right) = c\left(i,t'_{(k)}\right) e^{-\frac{t'_{(k)}}{\theta'}} \frac{\left(\frac{t'_{(k)}}{\theta'}\right)^{i}}{i!}.} \tag{13}$$

The acceptance probability is given by the same equation and taking t'_{A_i} in place of $t'_{(k)}$.

To calculate $c\left(i, t'_{(k)}\right)$ defined by (10) and (12) needs to enumerate all possible outcomes $(\delta_1, \delta_2, \ldots, \delta_k)$ which do not lead to a termination of the test before $t'_{(k)}$ and $\sum_{l=1}^{k} \delta_l = i$. Instead, we use the direct method of Aroian (Ref 11) to evaluate $\mathbb{P}\left(\left(i, t'_{(k)}\right); \theta'\right)$ for a given value of θ', say $\theta' = 1$, in order to get $c\left(i, t'_{(k)}\right)$.

To obtain $\mathbb{P}\left(\left(i, t'_{(k)}\right); \theta'\right)$ for the other values of θ', from equation (13), we only do variation of θ' because the coefficients $c\left(i, t'_{(k)}\right)$ do not depend on θ'.

4.1.2 Confidence limits after acceptance

We use the method described in paragraph 2 to evaluate the $100\,(1-\gamma)\,\%$ standardized confidence limits when the hypothesis H_0 has been accepted (the result of the sequential test leads to accept the equipment). Now, we use discrete variables. So the method allows us to reach conservative confidence limit.

As it can be seen on the Figure 4, the acceptance occurs at the standardized times $t'_{A_0}, t'_{A_1}, \ldots, t'_{A_{i_o-1}}$. On the vertical truncation, the standardized acceptance times are equal to t'_0.

In order to use the method described in paragraph 2, we need an estimator of θ'. Let T_A denotes the total time for acceptation.

The probability that the test ends at time t'_{A_i} is equal to the probability that the test ends with an acceptance of the equipment after i outages $\mathbb{P}\left(\left(i, t'_{A_i}\right); \theta'\right)$ and the probability that the test does not end with an acceptance is equal to $\mathbb{P}(\text{Rejection of the equipment}; \theta')$.

So T_A is a discrete random variable, which has the value t'_{A_i} for $i = 0, \ldots, i_0-1$, with probabilities $\mathbb{P}\left(\left(i, t'_{A_i}\right); \theta'\right)$ when the test ends with acceptance and the value $+\infty$ with probability $\mathbb{P}(\text{Rejection of the equipment}; \theta')$ when the test ends with rejection.

To estimate θ', we take

$$\hat{\theta}' = \frac{T_A}{N\left(T_A\right)} \mathbb{1}_{\{T_A < +\infty\}} \quad .$$

In this case we estimate θ' by $\frac{t'_{A_i}}{i}$ when the test ends at time t'_{A_i} with acceptance after i outages, and by 0 elsewhere.

The confidence limit that we obtain relies on the result of the test. We assume that **the test ends after i outages at time t'_{A_i} with acceptance**. The value of $\hat{\theta}'$ is equal to $\frac{t'_{A_i}}{i}$.

Let $\theta'_{L,\gamma,i}$ denotes the $100\,(1-\gamma)\,\%$ lower confidence limit of θ' when the test ends with acceptance after i outages. The $100\,(1-\gamma)\,\%$ lower confidence limit of θ $(\theta_{L,\gamma,i})$ is equal to $\theta_1\theta'_{L,\gamma,i}$.

The $100\,(1-\gamma)\,\%$ lower confidence value (see equation (9)) verifies the next equation

$$\sum_{s/\frac{t'_{A_s}}{s}\geq\frac{t'_{A_i}}{i}} \mathbb{P}\left(\hat{\theta}'=\frac{t'_{A_s}}{s};\theta'_{L,\gamma,i}\right)+\mathbb{P}\left(\hat{\theta}'=0;\theta'_{L,\gamma,i}\right)\mathbb{1}_{\left\{0\geq\frac{t'_{A_i}}{i}\right\}}=\gamma. \tag{14}$$

On the one hand, we have $\frac{t'_{A_i}}{i}>0$, hence the second part of the equation is equal to 0.

On the other hand, the sequence $\left(\frac{t'_{A_s}}{s}\right)_s$ is decreasing as we can see below.

- Let t'_{A_s} and $t'_{A_{s+1}}$ be two standardized acceptance times on the acceptance boundary before truncation, then
$$\frac{t'_{A_s}}{s}-\frac{t'_{A_{s+1}}}{s+1}=h'_0\left(\frac{1}{s}-\frac{1}{s+1}\right)>0 \;\text{ because } h'_0=\frac{-\log B}{1-1/d}>0.$$
- Let (s,t'_0) and $(s+1,t'_0)$ be two points of the acceptance boundary at the truncation such that $t'_{A_s}=t'_{A_{s+1}}=t'_0$, then
$$\frac{t'_{A_s}}{s}-\frac{t'_{A_{s+1}}}{s+1}=\frac{t'_0}{s}-\frac{t'_0}{s+1}>0.$$
- Let t'_{A_s} and $t'_{A_{s+1}}$ be two standardized acceptance time such that $t'_{A_{s+1}}=t'_0$ and t'_{A_s} is on the acceptance boundary before the truncation, then
$$\frac{t'_{A_s}}{s}-\frac{t'_{A_{s+1}}}{s+1}=s'+\frac{h'_0}{s}-\frac{t'_{A_{s+1}}}{s+1}\geq h'_0\left(\frac{1}{s}-\frac{1}{s+1}\right)>0 \;\text{ because } t'_{A_{s+1}}\leq(s+1)\,s'+h'_0.$$

Finally, we have $\left\{\hat{\theta}'=\frac{t'_{A_s}}{s}\right\}=\{\text{acceptance of the test with } s \text{ outages}\}$ so

$$\mathbb{P}\left(\hat{\theta}'=\frac{t'_{A_s}}{s};\theta'_{L,\gamma,i}\right)=\mathbb{P}\left(\left(s,t'_{A_s}\right);\theta'_{L,\gamma,i}\right).$$

We can rewrite equation (14) as the following

$$\gamma=\sum_{s=0}^{i}\mathbb{P}\left(\left(s,t'_{A_s}\right);\theta'_{L,\gamma,i}\right)$$

$$\boxed{\gamma=\sum_{s=0}^{i}c\left(s,t'_{A_s}\right)e^{-\frac{t'_{A_s}}{\theta'_{L,\gamma,i}}}\frac{\left(\frac{t'_{A_s}}{\theta'_{L,\gamma,i}}\right)^s}{s!}.} \tag{15}$$

We solve this last equation with respect to $\theta'_{L,\gamma,i}$ by the **bisection method** (ref 13).

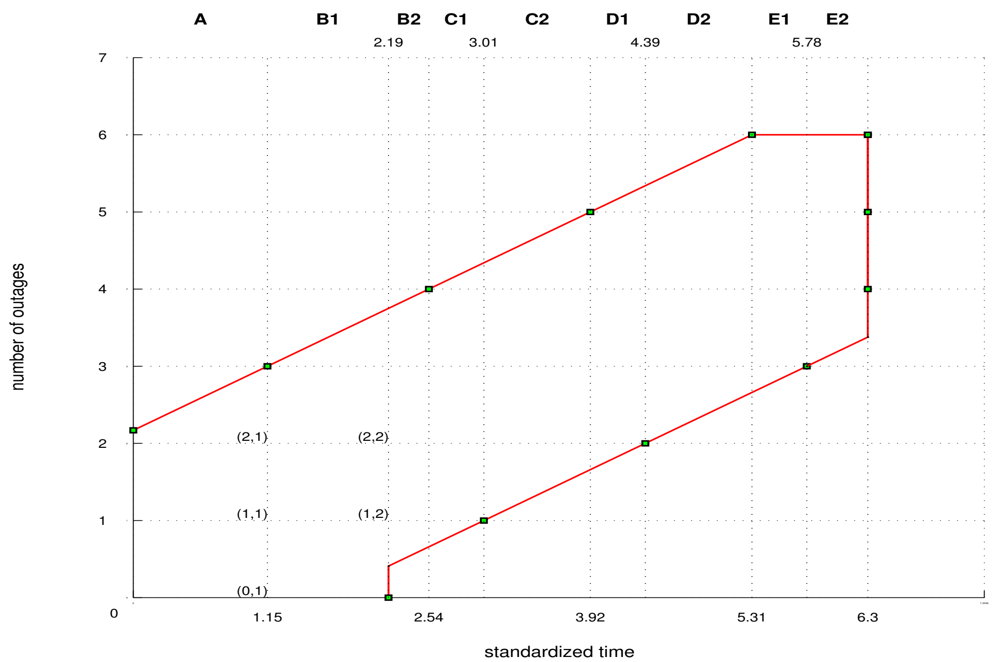


Figure 5: Homogeneity areas and numbering of continuation zones (number of outages, standardized times)

5 Calculations

5.1 Confidence interval

First we use the direct method of Aroian to calculate the sequence of the acceptance, continuation and rejection probabilities for the value $\theta' = 1$ of the standardized parameter.

This method calculates the acceptance, continuation and rejection probabilities step by step by the following way :

The standardized times associated with the intersections between the lines of outages and the acceptance boundaries or the rejection boundaries are calculated.

So, for $\alpha = 0,1$, $\beta = 0,4$, $d = 2$, $\theta_1 = 4000$ and a minimum observation time of one year, we obtain the sequence of standardized times of acceptance and rejection, which are

1,1507 2,19 2,5370 3,0082 3,9233 4,3945 5,3096 5,7808 6,3038

On the figure 5, the values of the rejection standardized times are rounded and written at the bottom. The values of the acceptance standardized times are written at the top.

We call homogeneity zone : a zone where the number of outages leading to rejection has a constant value $(A, B, \ldots, E)$.

The zone A extends from 0 to 1,1507 units of time: during this time a minimum of 3 outages is necessary to reject the equipment .

Inside zone B it is necessary to observe 4 outages at least for rejection. Zone B is composed of two zones: B_1 and B_2 . In zone B_1 acceptance occurs only with no outage. And so on,...

For each standardized time of acceptance or rejection, we calculate the acceptance and continuation probabilities $\mathbb{P}((i,t')\,;1)$ from which we obtain the values of the acceptance and continuation coefficients.

We use the so-called direct method of Aroian.

For each step m of the sequential test process, we define three zones

- D_m^1 the rejection zone,
- D_m^0 the acceptance zone
- D_m the continuation zone.

If $(X_1, \ldots, X_m)$ is a sample associated to D_m, we have

$$\mathbb{P}_\theta\left((X_1,\ldots,X_m)\in D_m\right)+\sum_{n=1}^{m}\mathbb{P}_\theta\left((X_1,\ldots,X_n)\in D_n^0\right)+\sum_{n=1}^{m}\mathbb{P}_\theta\left((X_1,\ldots,X_n)\in D_n^1\right)=1.$$

Denote
$$\begin{aligned} C_m &= \mathbb{P}_\theta\left((X_1,\ldots,X_m)\in D_m\right)\\ \mathbb{P}_m^0 &= \mathbb{P}_\theta\left((X_1,\ldots,X_m)\in D_m^0\right)\\ \mathbb{P}_m^1 &= \mathbb{P}_\theta\left((X_1,\ldots,X_m)\in D_m^1\right)\end{aligned}$$

At step 1, we have $C_1+\mathbb{P}_1^0+\mathbb{P}_1^1=1.$

At step 2, we have $C_2+\mathbb{P}_2^0+\mathbb{P}_2^1=C_1.$

Then at the last step, we simply have $\mathbb{P}_{m_0}^0+\mathbb{P}_{m_0}^1=C_{m_0-1}.$

The continuation probabilities at the time t'_{R_3} (=1,1507 : standardized rejection time after 3 outages at least) are given by

$$\begin{aligned}\mathbb{P}\left((0;1,1507)\,;1\right) &= e^{-1,1507}=0,3164\\ \mathbb{P}\left((1;1,1507)\,;1\right) &= 1,1507\cdot e^{-1,1507}=0,3641\\ \mathbb{P}\left((2;1,1507)\,;1\right) &= \frac{(1,1507)^2}{2}\cdot e^{-1,1507}=0,2095.\end{aligned}$$

These values are the probabilities that there are 0, 1 or 2 outages respectively during the standardized time 1,1507, when the true value of the parameter θ' is equal to 1. To evaluate these values, we use the Poisson distribution.

So, we now calculate $\mathbb{P}\left(\left(3,t'_{R_3}\right);1\right)$ as the complementary to 1 of the sum of the previous three probabilities.

Now, we explain how to calculate the continuation probabilities at the standardized time t'_{A_1} (=2,19).

To arrive at point (1,2) we have to

- start at (1,1) and detect no outage

or

- start at (0,1) and detect 1 outage.

The continuation probability calculated at (1,2) is

$$e^{(2,19-1,1507)}\left(0,3641+(2,19-1,1507)\times 0,3164\right)$$

and so on.

The acceptance probability at time t'_{A_1} (= 2,19) is

$$\mathbb{P}\left((0;2,19);1\right)=e^{-2,19}=0,1119.$$

This value is the probability that 0 outage occurs during the time 2,19, when the true value of the parameter θ' is equal to 1.

Then we get the values of acceptation coefficients $c\left(i,t'_{A_i}\right)$ and the values of continuation coefficients $c\left(i,t'\right)$ with the formulas

$$\begin{aligned} c\left(i,t'_{A_i}\right) &= e^{t'_{A_i}}\frac{i!}{\left(t'_{A_i}\right)^i}\mathbb{P}\left(\left(i,t'_{A_i}\right);1\right), \\ c\left(i,t'\right) &= e^{t'}\frac{i!}{\left(t'\right)^i}\mathbb{P}\left(\left(i,t'\right);1\right) \quad . \end{aligned}$$

So, we now are able to solve equation (15) with the bisection method, which gives the $100\left(1-\gamma\right)\%$ lower standardized limit when the test ends with acceptation.

5.2 Confidence probability

Now, it is possible to inverse the problem : when we know that the test ends with acceptance, we want to know the confidence level that the true MTBO is higher than the objective MTBO θ_1.

This means that we wish to calculate $\mathbb{P}(\theta'>1 \mid (i,t_{A_i}))$ which is the probability that the standardized true MTBO is higher than the standardized value 1 when the test ends with acceptance after i outages

This problem is the dual problem of the previous one. We use the same equation (15), but we solve it with respect to γ instead of θ'.

It is a sensitive problem to explain the meaning of this probability. The right way is to return to the first problem. Assume that the solution of equation (15) gives the value γ_0. Then we have to interpret the answer in the following way:

if we had wanted to search the $100\,(1-\gamma)\,\%$ *lower confidence boundary (*$\theta'_{L,\gamma_0,i}$*) we would have found 1.*

As here $\theta'_{L,\gamma,i}$ is fixed, then γ_0 varies as a function of i.

The next tables describe the obtained values. Many comparisons are made.

First, we present the graphs associated to the different studied plans with $\beta = 0,1$ to $0,4$ and $\alpha = 0,1$. Several configurations are detailed when $\beta = 0,4$.

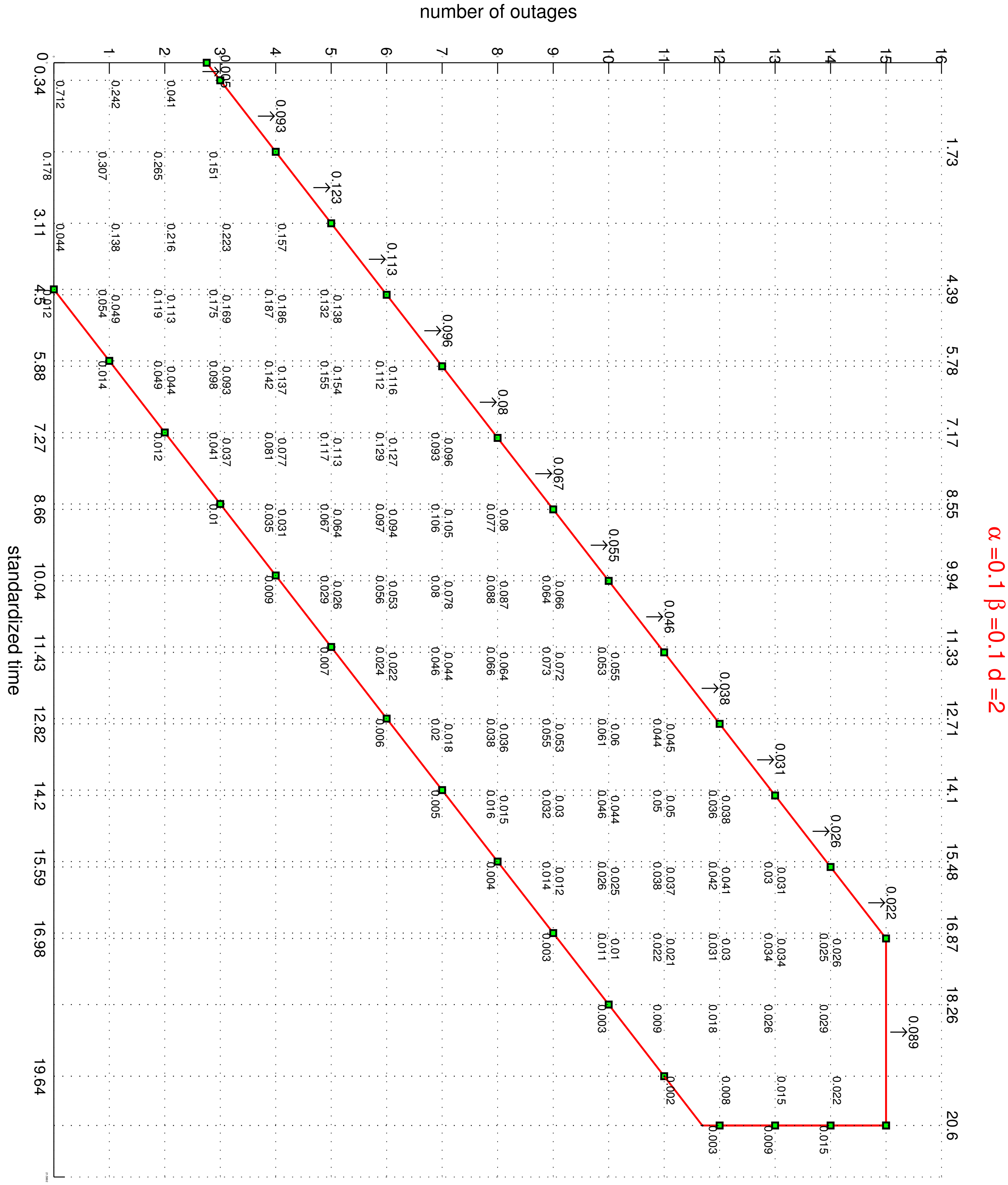
On the x-axis the numerical values of the standardized times are given.

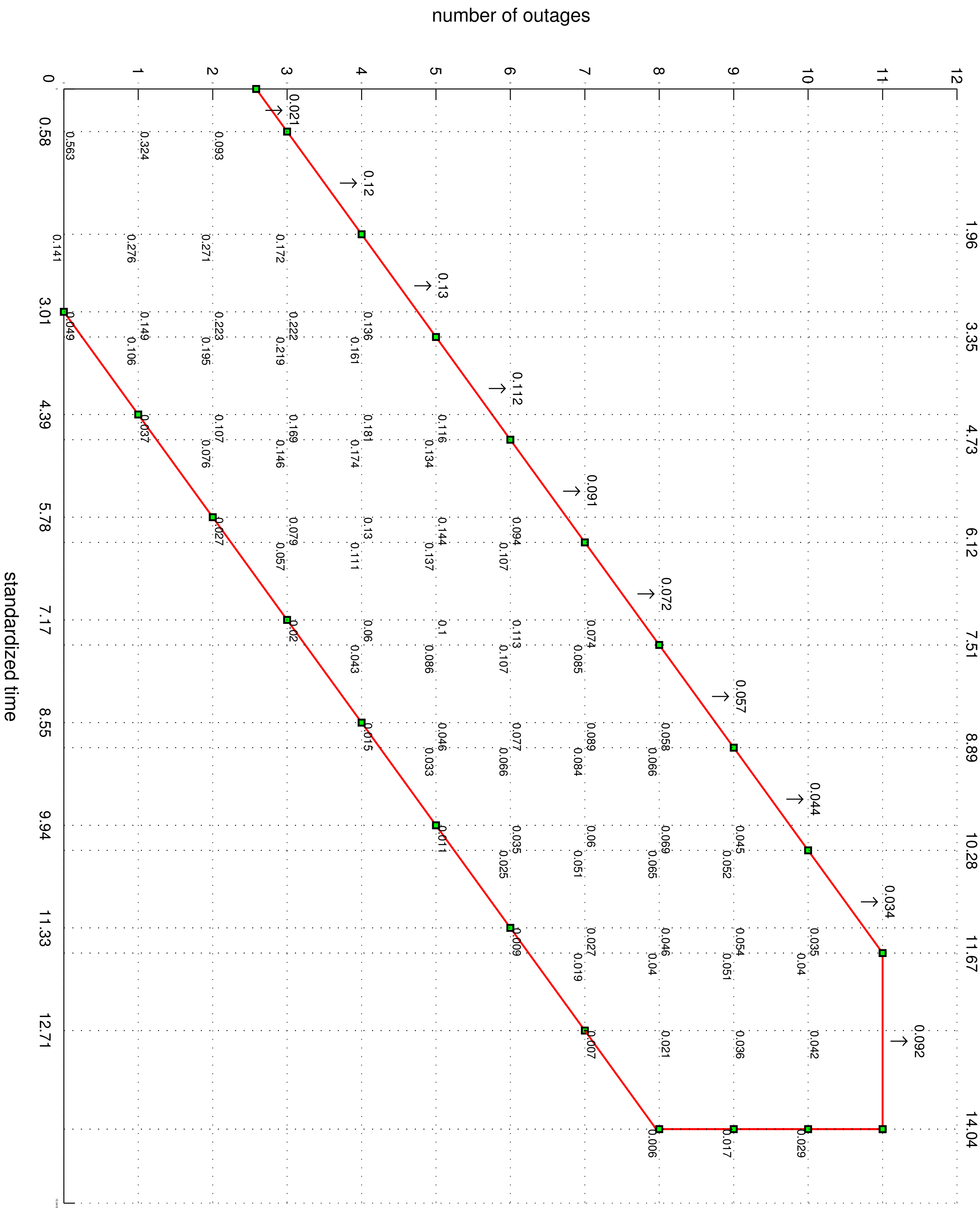
The continuation, acceptance and rejection probabilities are given, for each plan (up to 10^{-3}).

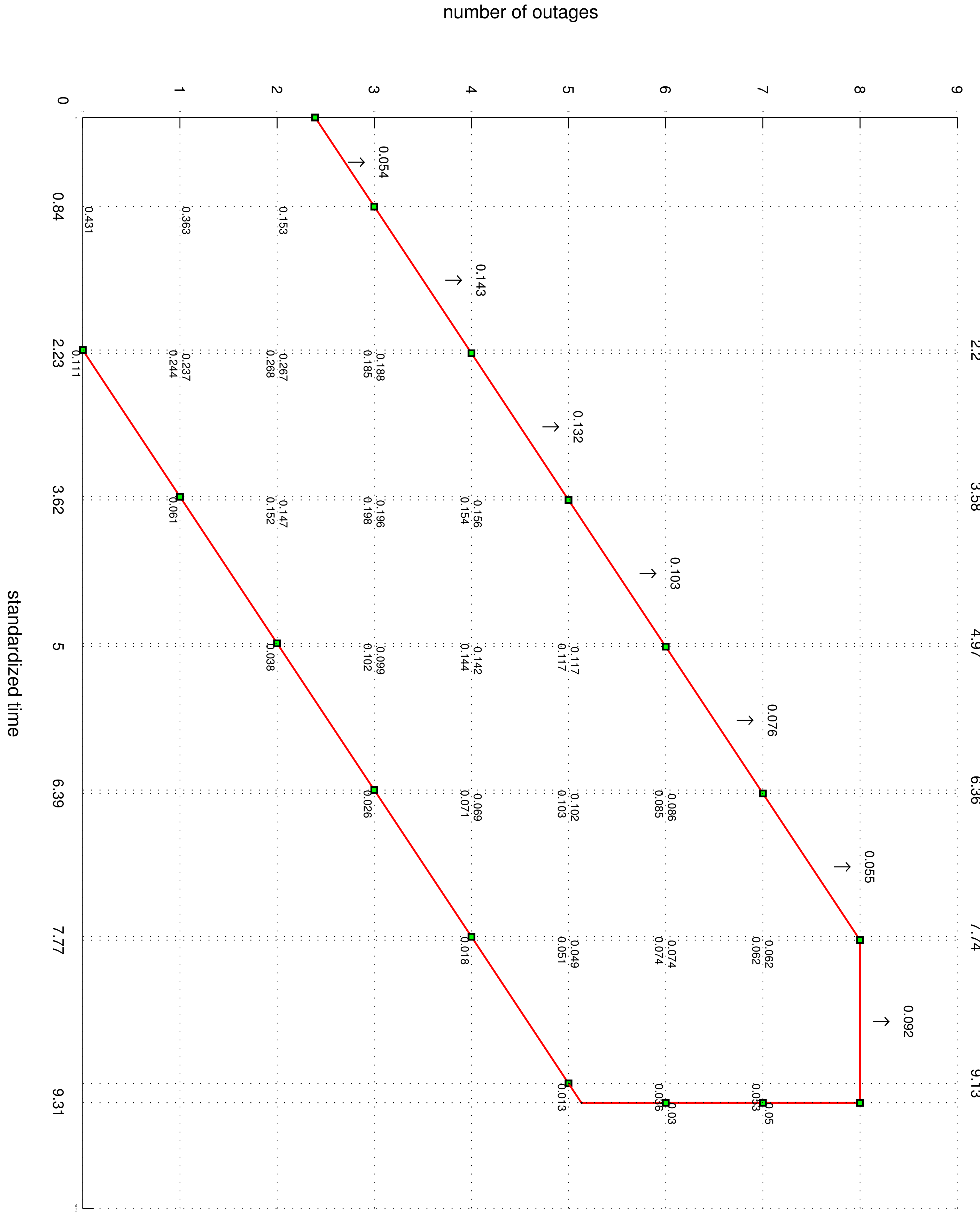
These probabilities allow us to solve equation (15) with respect to γ and to obtain the tables which also give the standardized times of observation.

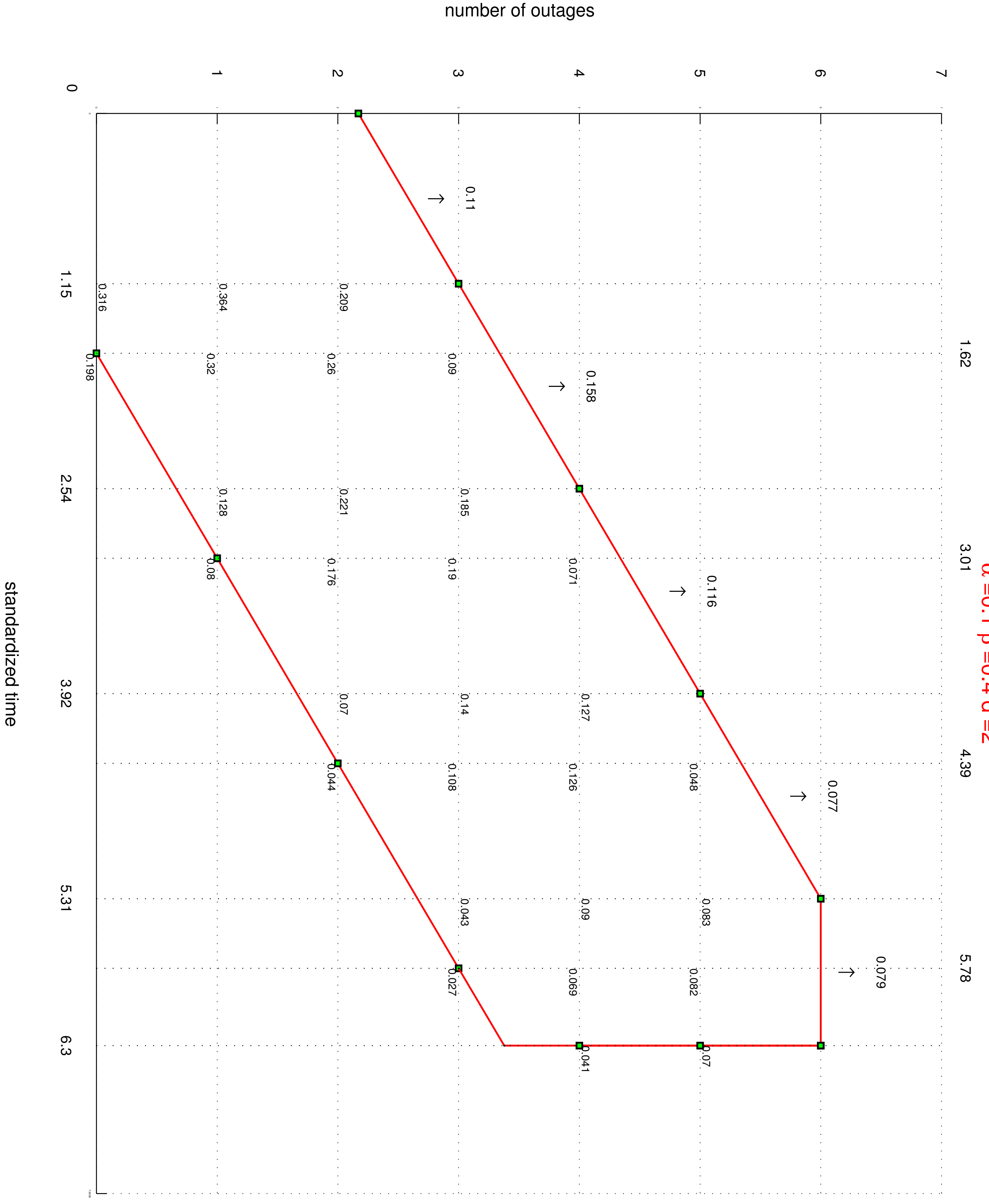
For an acceptation with 0 outage, the confidence probability varies from 95,06 when $\beta = 0,1$ to 88,81 when $\beta = 0,4$ under the condition that the minimum standardized observation time is equal to 2,19. If $\theta_1 = 4000$ hours, the minimum observation time is about equal to 1 year.

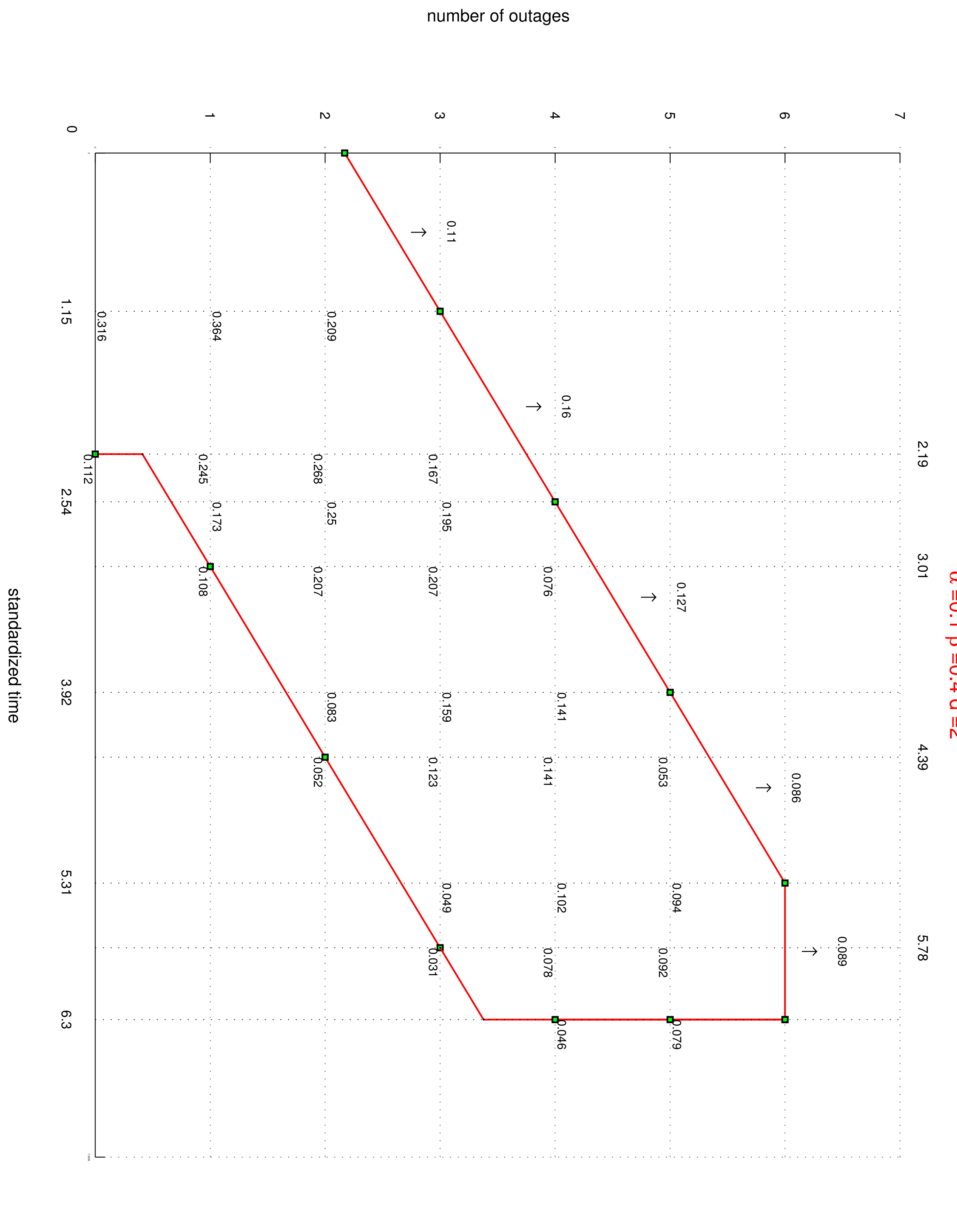
On the last table, we note that the confidence probabilities are always higher than 60%.

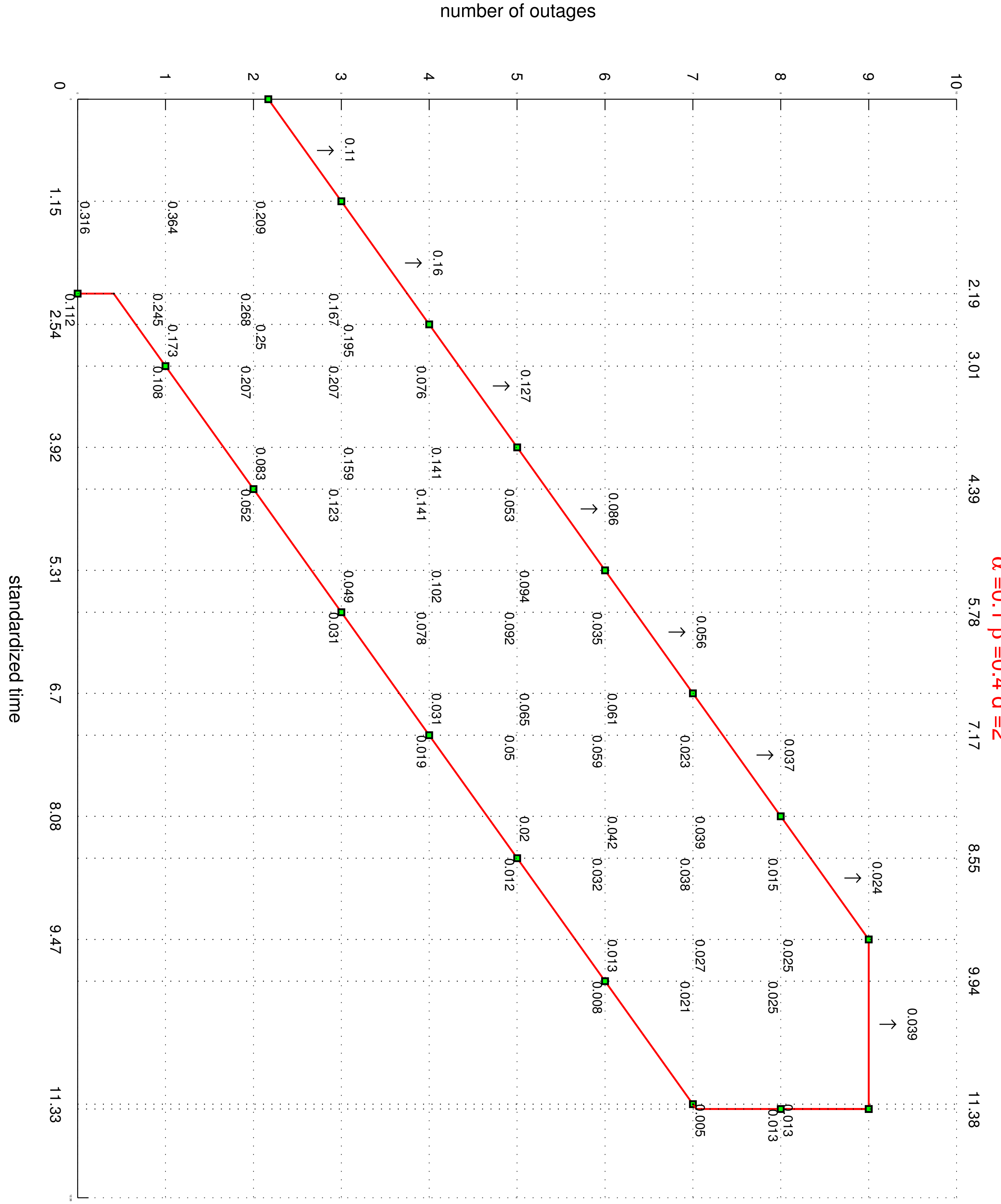

α =0.1 β=0.1 d =2
number of outages
standardized time


number of outages
standardized time


number of outages
standardized time
0
0.84
2.23
3.62
5
6.39
7.77
9.31
2.2
3.58
4.97
6.36
7.74
9.13
0.054
0.143
0.132
0.103
0.076
0.055
0.092


number of outages
standardized time


number of outages
standardized time


number of outages
standardized time

	Beta value					
	0.1 First plan[1]	0.2 First plan	0.3 First plan	0.4 First plan	0.4 Second plan[2]	0.4 Third plan[3]
0	98,77	95,06	88,89	80,25	88,81	88,81
t'_{A_0}	4,3944	3,0082	2,1972	1,6219	2,19	2,19
1	97,41	91,35	82,79	72,24	77,99	77,99
t'_{A_1}	5,7807	4,3944	3,5835	3,0082	3,0082	3,0082
2	96,19	88,66	78,99	67,84	72,82	72,82
t'_{A_2}	7,167	5,7807	4,9698	4,3944	4,3944	4,3944
3	95,16	86,68	76,44	65,13	69,73	69,73
t'_{A_3}	8,5533	7,167	6,3561	5,7807	5,7807	5,7807
4	94,3	85,18	74,66	61,04	65,12	67,79
t'_{A_4}	9,9396	8,5533	7,7424	6,3038	6,3038	7,167
5	93,58	84,04	73,4	54,06	57,26	66,54
t'_{A_5}	11,3259	9,9396	9,1287	6,3038	6,3038	8,5533
6	92,98	83,16	70,39			65,73
t'_{A_6}	12,7122	11,3259	9,3122			9,9396
7	92,48	82,48	65,4			65,22
t'_{A_7}	14,0985	12,7122	9,3122			11,3259
8	92,07	81,92				63,9
t'_{A_8}	15,4848	14,0415				11,3375
9	91,73	80,22				
t'_{A_9}	16,8711	14,0415				

TAB.1 - Confidence probabilities that the true MTBO > objective MTBO and associated standardized times as function of β when $\theta_0 = 8000$.

On this table, we can see that in the case of equipment acceptance after 0 outage with a minimum observation time equal to 1 year (modified plan) and $\beta = 0,4$, then the confidence probability that the true MTBO > 4000 hours is 88,81%.

With the same plan, we can see that in the case of equipment acceptance after 4 outages during $7,167 \times 4000$ hours, then the confidence probability that the true MTBO > 4000 hours is 67,79%.

[1] without minimum observation time

[2] the minimum observation time is equal to 1 year

[3] the maximum number of outages i_0 and the maximum observation time t_0 have been increased.

	Beta value			
	0.1 First plan	0.2 Second plan	0.3 Second plan	0.4 Second plan
0	98,77	98,75	98,75	98,75
t'_{A_0}	4,3944	4,38	4,38	4,38
1	97,41	93,34	93,26	93,26
t'_{A_1}	5,7807	4,3944	4,38	4,38
2	96,19	90,36	86,6	81,42
t'_{A_2}	7,167	5,7807	4,9698	4,3944
3	95,16	88,24	83,2	77,13
t'_{A_3}	8,5533	7,167	6,3561	5,7807
4	94,3	86,66	80,99	71,17
t'_{A_4}	9,9396	8,5533	7,7424	6,3038
5	93,58	85,46	79,45	61,33
t'_{A_5}	11,3259	9,9396	9,1287	6,3038
6	92,98	84,55	75,85	
t'_{A_6}	12,7122	11,3259	9,3122	
7	92,48	83,84	69,91	
t'_{A_7}	14,0985	12,7122	9,3122	
8	92,07	83,26		
t'_{A_8}	15,4848	14,0415		
9	91,73	81,5		
t'_{A_9}	16,8711	14,0415		

TAB.2 - Confidence probabilities that the true MTBO > objective MTBO and associated standardized times as function of β when $\theta_0 = 4000$.

6 Conclusion

The calculations done in the DERA report, are based on published and trusted theoretical results. Up to a good understanding of what a confidence interval is and what a confidence probability is, the suggestions appear to be consistent. Note that when we wish for a 4000 hours minimal operation time equipment, we choose a lower test MTBO equal to 4000 (θ_1) and fix θ_0 to 8000 hours. Doing this, we adopt a security point of view. Moreover, it has been advised to use the 60% confidence level only when there are evident reasons, before the test, to think that the equipment is good.

The fact that it is possible to take a decision with a few observations only, could be a surprise. Theoretically, this fact is true, under the conditions that the two statistical assumptions below are true. These two assumptions are

(i) the time between two outages has an exponential distribution,

(ii) the MTBO of the equipment is constant during the time.

In order to validate assumption (i) we need many observations. The validation of these two assumptions is of primordial importance, in order that the previous calculations can be trusted.

Note that it would be interesting to associate a prior probability to the possible value of the MTBO, instead of choosing between the fixed values θ_1 and θ_0 (Bayesian theory), then to calculate the posterior probability and to compare with the preceding results.

Note that this study does not address the case of several equipments operating separately with similar operational conditions. That also could decrease the qualification time.